Reduced notation, inner plethysms and the symmetric group

1993 J. Phys. A: Math. Gen. 26 7461

(http://iopscience.iop.org/0305-4470/26/24/019)

J. Phys. A: Math. Gen. 26 (1993) 7461–7478. Printed in the UK

Reduced notation, inner plethysms and the symmetric group

Thomas Scharf†‖, Jean-Yves Thibon‡¶ and Brian G Wybourne§††

† Lehrstuhl II für Mathematik, Universität Bayreuth, D-95440 Bayreuth, Federal Republic of Germany

‡ Institut Gaspard Monge, Université de Marne-la-Vallée, 2 rue de la Butte-Verte, 93166 Noisy-le-Grand Cedex, France

§ Instytut Fizyki, Uniwersytet Mikołaja Kopernika, ul. Grudziądzka 5/7, 87-100 Toruń, Poland

Received 12 August 1993

Abstract. The reduced notation for irreducible representations of the symmetric group S_n is interpreted in terms of symmetric formal series and vertex operators, and is used to prove a number of properties of reduced Kronecker products and inner plethysms in an n-independent manner. Conditions for self-associativity of Kronecker products and inner plethysms are established. Reduced inner plethysms are developed and applied to the question of non-simple phase groups among the symmetric S_n and alternating A_n groups.

1. Introduction

The symmetric group S_n plays an important role in all those areas of physics and chemistry involving permutational symmetry. These include high-symmetry molecules, symplectic models of nuclei and such esoteric topics as the classification of N-electron states of quantum dots. It is highly desirable to be able to develop stable results that are essentially n-independent. This is made possible by exploiting the reduced notation introduced long ago (Murnaghan 1938, Littlewood 1958a, b) and only recently made mathematically precise (Thibon 1991). The reduced notation supplies a relatively concise method of evaluating n-independent Kronecker products for the symmetric group (Butler and King 1973, Thibon 1991). Extensions to Kronecker products involving projective (or spin) representations of the symmetric group have also been made (Luan Dehuai and Wybourne 1981, Salam and Wybourne 1989).

Symmetrized Kronecker powers of representations of the symmetric group play an important role in nuclear physics applications (Kretzschmar 1960a, b, Vanagas 1971), in determining the symmetry properties of $3j$ symbols for the symmetric group and its subgroups (Butler 1974, King 1974) and in determining branching rules for Lie groups where the symmetric group occurs as a finite subgroup (Salam and Wybourne 1989).

In all the above problems Schur functions (S-functions) play a key role: for the Kronecker products, in the form of the Littlewood–Richardson rule for multiplying S-functions (Littlewood and Richardson 1934, Littlewood 1950) and in symmetrized powers, the inner plethysm of S-functions (Littlewood 1958a, b). For a modern account of S-functions the reader is referred to Macdonald (1979) and for inner plethysm in a Hopf

‖ Partially supported by NATO scientific board via DAAD.

¶ Supported by PRC Math-Info.

†† Supported by Polish KBN PB573/2/91.

algebra structure to Scharf and Thibon (1992) and to Thibon (1991). Much of our notation is drawn from those sources and the original papers of Littlewood.

In this paper we first address the problem of reduced notation in calculating Kronecker products for the symmetric group noting in particular the conditions for the resultant of a reduced Kronecker product to be self-associated. We then show how use of the reduced notation for Kronecker products can be used to uncover hitherto unnoticed unimodal distributions of multiplicities and to exhibit certain symmetries. We next show that reduced Kronecker products $\langle\lambda\rangle * \langle\mu\rangle$ are self-associated if one of the partitions is a staircase partition and the other partition is at least self-conjugate. After that we take up the problem of evaluating reduced inner plethysms of S-functions and outline methods for their evaluation. We observe that certain reduced inner plethysms are self-associated. The reduced inner plethysms are then used to establish that for $n \geqslant 6$ the symmetric groups $\mathcal{S}_n$ are all non-simple phase groups (Derome 1966, van Zanten and de Vries 1973). Correspondingly we show that the alternating groups $\mathcal{A}_n$ with $n \geqslant 7$ are also non-simple phase groups. An expansion of the reduced S-function $\langle n\rangle$ as a linear combination of inner plethysms involving only S-functions indexed by single hook partitions is next proved and a simple proof of the Butler–Boorman theorem established. Finally, we discuss a number of special properties of S-functions indexed by staircase partitions and prove earlier inferred results.

A remark about notation. In this paper, we shall freely mix the traditional notation of Littlewood (1950) with that of Macdonald (1979), which is more convenient for algebraic manipulations. So, the S-function corresponding to a partition λ will be indifferently denoted by $\{\lambda\}$ or by s_λ. Other correspondences between the two systems will be explained when necessary.

2. Kronecker products

The resolution of the *inner* or *internal product* of two S-functions, say $s_\lambda * s_\mu$ with $\lambda, \mu \vdash n$ such that

$$s_\lambda * s_\mu = \sum_{\nu\vdash n} c^\nu_{\lambda\mu} s_\nu \tag{1}$$

is related to the character analysis for the symmetric group $\mathcal{S}_n$ where

$$\chi^{(\lambda)}_\rho \chi^{(\nu)}_\rho = \sum_\nu c^\nu_{\lambda\mu} \chi^{(\nu)}_\rho \tag{2}$$

with the coefficients $c^\nu_{\lambda\mu}$ being non-negative integers and precisely the same integers that arise in the resolution of the Kronecker product of two irreducible representations of $\mathcal{S}_n$ labelled by the same pair of partitions. Much effort has gone into the development of algorithms for calculating the coefficients $c^\nu_{\lambda\mu}$ (cf Murnaghan 1938, Littlewood 1958a, b, Butler and King 1973, Thibon 1991). Remmel (1989) has presented a formula for the Kronecker products of S-functions of hook shapes. His work has recently been extended (Remmel and Whitehead 1993) to include two-row shapes. This recent work may be substantially simplified and the results cast in an n-independent form using the reduced notation (Murnaghan 1938).

3. Reduced notation, symmetric series and vertex operators

The concept of reduced notation for the symmetric group was introduced by Murnaghan (1938) and later used (Littlewood 1958a, b) for the calculation of inner plethysms and Kronecker products for the symmetric group $\mathcal{S}_n$. The significance of the reduced notation was further emphasized by Butler and King (1973). Extensions to the projective (or spin) representations have been discussed (Luan Dehuai and Wybourne 1981, Salam and Wybourne 1989). A rigorous treatment in terms of the Hopf algebras of symmetric functions has been given for the case of tensor products of symmetric group representations (Thibon 1991) and will be outlined below.

The tensor irreducible representation $\{\lambda\}$ of $\mathcal{S}_n$ may be labelled by ordered partitions (λ) of integers where $\lambda \vdash n$. In reduced notation the label $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ for $\mathcal{S}_n$ is replaced by $\langle\lambda\rangle = \langle\lambda_2, \ldots, \lambda_p\rangle$. Given any irreducible representation $\langle\mu\rangle$ in reduced notation it can be converted back into a standard irreducible representation of $\mathcal{S}_n$ by prefixing it with a part $(n - |\mu|)$. For example, an irreducible representation $\langle 21\rangle$ in reduced notation corresponds to $\{321\}$ in $\mathcal{S}_6$ or $\{921\}$ in $\mathcal{S}_{12}$. It is just this feature that leads to an n-independent notation for $\mathcal{S}_n$. If $n - |\mu| \geqslant \mu_1$ then the resulting irreducible representation $\{n - |\mu|\}$ is assuredly a standard irreducible representation of $\mathcal{S}_n$. However, if $n - |\mu| < \mu_1$ then the irreducible representation $\{n - |\mu|, \mu\}$ is non-standard and must be converted into standard form using S-function modification rules (Littlewood 1950). Thus in $\mathcal{S}_3$ $\langle 21\rangle$ becomes $-\{1^3\}$ and in $\mathcal{S}_4$ is null.

The reduced label $\langle\mu\rangle$ can be given a precise mathematical meaning: it has to be interpreted as the formal sum

$$\langle\mu\rangle = \sum_{n=-\infty}^{\infty} s_{(n,\mu)} \tag{3}$$

(the non-standard S-function $s_{(n,\mu)} = \{n, \mu\}$ being zero for $n \ll 0$). The advantage of this point of view is that this series can be obtained by applying to s_μ a certain infinite-order differential operator, which is one of the so-called *vertex operators* commonly used in the representation theory of infinite-dimensional Lie algebras (see e.g. Kac 1990). Vertex operators also exist for S-functions, Q-functions and Hall–Littlewood functions (Hoffman 1989, Jarvis and Yung 1993, Jing 1991a, b, Salam and Wybourne 1991, 1992). The vertex operator has certain remarkable properties which makes it a very convenient tool for formal manipulations in reduced notation. We would hope the reader does not, in the following text, confuse the subscripted λ_z of standard λ-ring notation with the use of λ for partitions introduced earlier. One has

$$\langle\mu\rangle = \Gamma_1\, s_\mu \tag{4}$$

where (with a parameter z in order to have more generally $\sum_n s_{(n,\mu)} z^n = \Gamma_z s_\mu$)

$$\Gamma_z = \exp\left\{\sum_{k\geqslant 1} z^k \frac{p_k}{k}\right\} \exp\left\{-\sum_{k\geqslant 1} z^{-k} \frac{\partial}{\partial p_k}\right\} \tag{5}$$

where p_k is the kth power-sum symmetric function. Introducing the generating series for the complete and elementary symmetric functions

$$\sigma_z = \sum_{n\geqslant 0} h_n z^n \qquad \lambda_z = \sum_{n\geqslant 0} e_n z^n$$

and the 'Foulkes derivative', defined by

$$(D_F G, H) = (G, FH)$$

where F, G, H are any three symmetric functions and $(\ ,\ )$ is the usual scalar product defined by $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$, one obtains for Γ_z the following convenient expression

$$\Gamma_z = \sigma_z\ D_{\lambda_{-1/z}} \tag{6}$$

(see e.g. Carré and Thibon 1992). To interface between the reduced notation and the operator notation, we shall set, for a symmetric function F, $\langle F\rangle = \Gamma_1 F$, so that for example $\langle\lambda\rangle = \langle s_\lambda\rangle$. When F is a positive sum of S-functions, we will call the series $\langle F\rangle$ a *stable character*.

Our symmetric functions are supposed to be polynomials in some (infinite) set of variables $X = \{x_1, x_2, \ldots\}$, usually called 'the alphabet'. The effect of the operator $D_{\sigma_z(X)}$ on a symmetric function $F(X)$ is to replace X by $X + z$, that is to replace $p_k(X)$ by $p_k(X + z) = p_k(X) + z^k$. Similarly $D_{\lambda_{-z}(X)}$ acts by replacing X by $X - z$, that is by substituting $p_k(X)$ by $p_k(X) - z^k$. Thus, D_{σ_z} is invertible, and $D_{\sigma_z}^{-1} = D_{\lambda_{-z}}$. The operators are even multiplicative with respect to Littlewood–Richardson multiplication. It should be noted that they fulfil the above equation for the Foulkes operators with $F := \sigma_z, \lambda_{-z}$.

The notations $X + z, X - z$ are particular instances of the so-called λ-ring formalism. The concept of a λ-ring, which is due to Grothendieck, is explained in Knutson (1972) and in Lascoux and Pragacz (1988). It provides a convenient way to handle what Littlewood called the S-functions of an arbitrary series, by allowing the coefficients of any power series to be considered as the homogenous (or elementary) functions of some virtual alphabet.

One example for the application of vertex operators is Gamba's (1952) formula for character polynomials Ξ^λ (cf Kerber 1992). These polynomials yield, evaluated at a (finite) sequence of non-negative integers $(a_1, \ldots, a_r)$, the character value $(\langle\lambda\rangle, p^a)$, $p^a := p_1^{a_1} \cdots p_r^{a_r}$, and hence also code irreducible characters in reduced notation. This n-independence of character values has been observed by Frobenius (1900, 1904) and has been investigated by Murnaghan (1938, 1951, 1955), Gamba (1952), Specht (1960), Wagner (1979) and others. Morris applied the concept to Schur Q-functions and Hall–Littlewood Q-functions (Morris 1963a, b).

Gamba's formula can be deduced in the following way

$$(\langle\lambda\rangle, p^a) = (\sigma_1 D_{\lambda_{-1}} s_\lambda, p^a)(D_{\lambda_{-1}} s_\lambda, D_{\sigma_1} p^a) = \left(D_{\lambda_{-1}} s_\lambda, \prod_i (p_i + 1)^{a_i}\right).$$

Obviously the last expression can be identified as the value of a polynomial $\Xi^\lambda(X_1, \ldots, X_s)$ in a finite number of variables evaluated at $X_i = a_i, i \leqslant s$, (and $a_i = 0$ for $i > r$).

Reduced notation refers to a stability property of the sequence $s_{(n,\lambda)}, n \in \mathbb{Z}$, of (up to coefficient $0, \pm 1$) irreducible characters. But the operator $D_{\lambda_{-1}}$ is an automorphism making homogeneous symmetric functions inhomogeneous. Also series of type

$$\langle\!\langle F\rangle\!\rangle := \sigma_1 F \tag{7}$$

are easier to handle and formulae involving them can be transformed to reduced notation using $D_{\sigma_1} D_{\lambda_{-1}} = 1$. This also reflects another type of stability displayed by the series $h_{(n,\lambda)}, n \in \mathbb{Z}$. The h_λ being the symmetric functions associated with permutation representation by the Frobenius correspondence, we will refer to series of the form $\langle\!\langle h_\lambda\rangle\!\rangle$ as *stable permutation characters*. In the language of character polynomials $\langle\!\langle h_\lambda\rangle\!\rangle$ correspond to the so-called Young polynomials (Specht 1960, Kerber 1992).

4. Reduced Kronecker products

A reduced Kronecker product $\langle\lambda\rangle * \langle\mu\rangle$ may be evaluated by the recursive relation (Littlewood 1958b, Butler and King 1973, Wagner 1979)

$$\langle\lambda\rangle * \langle\mu\rangle = \sum_{\alpha,\beta,\gamma} \langle\{\lambda/\alpha\gamma\} \cdot \{\mu/\beta\gamma\} \cdot \{\alpha * \beta\}\rangle \tag{8}$$

where / indicates an S-function skew (i.e. $\{\lambda/\mu\} = D_{s_\mu} s_\lambda$, cf Macdonald (1979)), a dot $\cdot$ is for Littlewood–Richardson S-function multiplication and a star $*$ is the ordinary inner product. For example, in reduced notation

$$\begin{aligned}\langle 21\rangle * \langle 2^2\rangle = {} & \langle 51\rangle + \langle 5\rangle + \langle 43\rangle + \langle 421\rangle + 3\langle 42\rangle + 3\langle 41^2\rangle + 5\langle 41\rangle + 3\langle 4\rangle + \langle 3^21\rangle + 2\langle 3^2\rangle \\ & + \langle 32^2\rangle + \langle 321^2\rangle + 6\langle 321\rangle + 7\langle 32\rangle + 3\langle 31^3\rangle + 8\langle 31^2\rangle + 8\langle 31\rangle + 3\langle 3\rangle + \langle 2^31\rangle \\ & + 2\langle 2^3\rangle + 3\langle 2^21^2\rangle + 7\langle 2^21 + 5\langle 2^2\rangle + \langle 21^4\rangle + 5\langle 21^3\rangle + 8\langle 21^2\rangle + 6\langle 21\rangle + 2\langle 2\rangle \\ & + \langle 1^5\rangle + 3\langle 1^4\rangle + 3\langle 1^3\rangle + 2\langle 1^2\rangle + \langle 1\rangle.\end{aligned} \tag{9}$$

Equation (8) may be deduced from a more general result involving stable permutation characters (Thibon 1991)

$$\langle\!\langle F\rangle\!\rangle * \langle\!\langle G\rangle\!\rangle = \left\langle\!\!\left\langle \sum_{\alpha,\beta} (D_{u_\alpha} F) \cdot (D_{u_\beta} G) \cdot (v_\alpha * v_\beta) \right\rangle\!\!\right\rangle \tag{10}$$

where $(u_\gamma), (v_\gamma)$ are adjoint bases of symmetric functions.

Another (almost direct) consequence of (10) is the following formula of Murnaghan

$$\langle F\rangle * \langle G\rangle = \left\langle \sum_\alpha (D_{p_\alpha} F \cdot D_{p_\alpha} G) \frac{D_{\sigma_1} p_\alpha}{(p_\alpha, p_\alpha)} \right\rangle. \tag{11}$$

Let us return to example (9). It reveals a hitherto unnoticed feature: that some reduced Kronecker products are self-associated in the sense that replacing every partition by its conjugate partition leaves the product invariant. We shall shortly establish the condition for a product to be self-associated.

We also observe that Kronecker products often contain sequences of certain types of partitions whose multiplicities have a unimodal distribution when these partitions are presented in reverse lexicographic order. This is the case, for example, of the terms of the type $\langle m, 1^2\rangle$ in the reduced product

$$\begin{aligned}\langle 14\rangle * \langle 10\rangle \supset {} & \langle 21\ 1^2\rangle + \langle 20\ 1^2\rangle + 2\langle 19\ 1^2\rangle + 2\langle 18\ 1^2\rangle + 3\langle 17\ 1^2\rangle + 3\langle 16\ 1^2\rangle + 4\langle 15\ 1^2\rangle + 4\langle 14\ 1^2\rangle \\ & + 5\langle 13\ 1^2\rangle + 4\langle 12\ 1^2\rangle + 4\langle 11\ 1^2\rangle + 3\langle 10\ 1^2\rangle + 3\langle 91^2\rangle + 2\langle 81^2\rangle \\ & + 2\langle 71^2\rangle + \langle 61^2\rangle + \langle 51^2\rangle.\end{aligned} \tag{12}$$

The above Kronecker product involves a total multiplicity sum of 1701 distributed over 377 distinct partitions effectively concealing unimodal sequences that also exhibit certain symmetries. These special distributions can be uncovered by developing explicit expressions for the coefficients $c^\nu_{\lambda\mu}$. We illustrate this in the next section.

5. Kronecker products for two-row shapes

In terms of reduced notation two-row shapes become one-row shapes via the equivalence

$$\{n-k, k\} * \{n-\ell, \ell\} \sim \langle k \rangle * \langle \ell \rangle. \tag{13}$$

Noting equation (8) we have

$$\begin{aligned} \langle k \rangle \langle \ell \rangle &= \sum_{q=0}^{p} \sum_{p=0}^{\ell} \langle \{k-p\} \cdot \{\ell - p\} \cdot \{p-q\} \rangle \\ &= \sum_{\lambda} c^{\nu}_{\lambda\mu} \langle \lambda \rangle \qquad \text{for } k \geqslant \ell. \end{aligned} \tag{14}$$

The possible shapes for λ are severely constrained. The number of rows cannot exceed three. The multiplicity to be associated with a given shape λ can be readily determined by drawing the shape and then filling the cells, in accordance with the Littlewood–Richardson rule, with say $k-p$ circles $\circ$, $\ell - p$ stars $\star$ and $p-q$ diamonds $\diamond$, where $k + \ell - p + q = \lambda_1 + \lambda_2 + \cdots + \lambda_p$. Repeated cells are marked with dots $\cdot$. Consider the shape characterized by the partition (m). A typical filling is shown below:

$$| \leftarrow k-p \rightarrow | \leftarrow \ell - p \rightarrow | \leftarrow p-q \rightarrow |$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \circ & \circ & \cdot & \cdot & \cdot & \circ & \star & \star & \cdot & \cdot & \cdot & \diamond & \diamond & \cdot & \cdot & \cdot & \diamond \\ \hline \end{array}$$

From which we may deduce immediately that $c^{(m)}_{\langle k \rangle \langle \ell \rangle}$ is the number of partitions of $k + \ell - m$ into two parts (p, q) with $p \geqslant q$ and $\ell \geqslant p$ leading to

$$c^{\langle m \rangle}_{\langle k \rangle \langle \ell \rangle} = \tfrac{1}{2}(\ell - k + m + 2) \qquad \text{for } k > m \tag{15a}$$

$$c^{\langle m \rangle}_{\langle k \rangle \langle \ell \rangle} = \tfrac{1}{2}(k + \ell - m + 2) \qquad \text{for } m \geqslant k \tag{15b}$$

and the coefficient symmetry

$$c^{\langle m \rangle}_{\langle k \rangle \langle \ell \rangle} = c^{\langle 2k-m \rangle}_{\langle k \rangle \langle \ell \rangle} \tag{15c}$$

($[x/2]$ is taken to be the integer part of the division by 2). In a similar manner we find

$$c^{\langle m,1 \rangle}_{\langle k \rangle \langle \ell \rangle} = \ell - k + m + 1 \qquad \text{for } k > m \tag{16a}$$

$$c^{\langle m,1 \rangle}_{\langle k \rangle \langle \ell \rangle} = k + \ell - m \qquad \text{for } m \geqslant k \tag{16b}$$

with

$$c^{\langle m,1 \rangle}_{\langle k \rangle \langle \ell \rangle} = c^{\langle 2k-m-1,1 \rangle}_{\langle k \rangle \langle \ell \rangle} \tag{16c}$$

and

$$c^{\langle m,1^2 \rangle}_{\langle k \rangle \langle \ell \rangle} = \tfrac{1}{2}(\ell - k + m + 1) \qquad \text{for } k > m + 1 \tag{17a}$$

$$c^{\langle m \rangle}_{\langle k \rangle \langle \ell \rangle} = \tfrac{1}{2}(k + \ell - m + 2) \qquad \text{for } m \geqslant k - 1 \tag{17b}$$

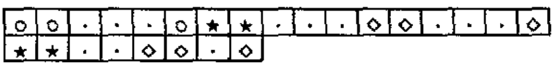
with

$$c^{(m,1^2)}_{(k)(\ell)} = c^{(2k-m-2,1^2)}_{(k)(\ell)}. \tag{17c}$$

Equations (15c), (16c) and (17c) reveal the unimodal multiplicity distribution of certain types of partitions when sequenced in reverse lexicographic order together with certain symmetries of the coefficients. Equations (17a)–(17c) give a simple explanation of the result of equation (12).

The above examples all involve simple two-part partition functions that admit a simple algebraic description. In more complex cases constrained partition functions arise with a series of subsidiary conditions. Thus to evaluate the multiplicity $c^{(m,n)}_{(k)(\ell)}$ one draws the shape for (m, n) as before to give

$$| \leftarrow k-p \rightarrow | \leftarrow \ell-p-r \rightarrow | \leftarrow p-q-s \rightarrow |$$



Inspection of the diagram leads to the conditions

$$n = r+s \qquad \ell - p \geqslant r \qquad \ell \geqslant p \qquad p-q \geqslant s \qquad \text{and } p+q = k+\ell-m-n. \tag{18a}$$

Furthermore,

$$\text{if} \quad r=0 \qquad \text{then} \qquad k+\ell-2p \geqslant n \tag{18b}$$

and hence the coefficients $c^{(m,n)}_{(k)(\ell)}$ may be evaluated by the following steps

(i) List the partitions (p, q) subject to $p \geqslant q$ and $\ell \geqslant p$.

(ii) Associate with each partition (p, q) a set of two part partitions (r, s) that are compatible with the conditions of equation (18).

(iii) $c^{(m,n)}_{(k)(\ell)}$ is equal to the total number of partitions (r, s) found in (ii).

Thus to evaluate $c^{(9,5)}_{(20)(16)}$ we obtain the values in table 1.

Table 1. Partitions (p, q) and (r, s) for evaluating $c^{(9,5)}_{(20)(16)}$.

(p,q)	Condition on r	Condition on s	(r,s)
(16, 6)	$0 \geqslant r = 0$	$10 \geqslant s$	*
(15, 7)	$1 \geqslant r \geqslant 0$	$8 \geqslant s$	(0, 5), (1, 4)
(14, 8)	$2 \geqslant r \geqslant 0$	$6 \geqslant s$	(0, 5), (1, 4), (2, 4)
(13, 9)	$3 \geqslant r \geqslant 0$	$4 \geqslant s$	(1, 4), (2, 3), (3, 2)
(12, 10)	$4 \geqslant r \geqslant 0$	$2 \geqslant s$	(4, 1), (3, 2)
(11, 11)	$5 \geqslant r \geqslant 0$	$0 \geqslant s$	(5, 0)

From these we deduce that $c^{(9,5)}_{(20)(16)} = 11$. Note that equation (18a) rules out any partition (r, s) for the first line in table 1 and that the partitions (r, s) are not restricted to $r \geqslant s$.

Table 2. Partitions (p, q) and (r, s) for evaluating $c_{(20)(16)}^{\langle 10,4,2\rangle}$.

(p, q)	Condition on r	Condition on s	(r, s)
(16, 4)	$0 \geqslant r \geqslant= 2$	$10 \geqslant s$	*
(15, 5)	$1 \geqslant r \geqslant 2$	$8 \geqslant s$	*
(14, 6)	$2 \geqslant r \geqslant 2$	$6 \geqslant s$	(2, 2)
(13, 7)	$3 \geqslant r \geqslant 2$	$4 \geqslant s$	(2, 2), (3, 1)
(12, 8)	$4 \geqslant r \geqslant 2$	$2 \geqslant s$	(4, 0), (3, 1), (2, 2)
(11, 9)	$5 \geqslant r \geqslant 2$	$0 \geqslant s$	(4, 0)
(10, 10)	$6 \geqslant r \geqslant 2$	$-2 \geqslant s$	*

The evaluation of the coefficients $c_{\langle k\rangle\langle \ell\rangle}^{\langle m,n,t\rangle}$ proceeds in an exactly similar manner. The diagram for the shape of (m, n, t) is drawn and the partition constraints determined to give

$$n = r + s \qquad \ell - p \geqslant r \geqslant t \qquad p - q \geqslant t + s \qquad p + q = k + \ell - m - n - t. \quad (19)$$

Thus to evaluate $c_{\langle 20\rangle\langle 16\rangle}^{\langle 10,4,2\rangle}$ we have the values in table 2, showing that $c_{\langle 20\rangle\langle 16\rangle}^{\langle 10,4,2\rangle} = 7$. The first two lines, and the last, could have been anticipated to yield a null result.

In one-row reduced notation the results (see equations (15*a*) and (15*b*)) are extremely simple, whereas for two- and three-row reduced notation there are subsidiary conditions to be considered. In most physical applications, due to spin considerations and the consequential Pauli exclusion principle, one-rowed reduced notation suffices.

6. Self-associated reduced Kronecker products

The self-associativity observed in equation (9) is a direct consequence of the following theorem.

Theorem 6.1. For H defined by $\langle\lambda\rangle * \langle\mu\rangle = \langle H\rangle$ to be self-associated, it is necessary and sufficient that one of the partitions be a staircase partition and the other partition be at least self-conjugate.

It is better to reformulate the properties. A symmetric function G is self-associated if and only if it can be written as a linear combination of products of symmetric power sums $p_\alpha := p_{\alpha_1} \cdots p_{\alpha_r}$ with $|\alpha| - \ell(\alpha)$ even. Indeed, consider the involution ω defined by $\omega s_\mu = s_{\mu'}$ (cf Macdonald 1979). Then, the self-associativity of G, that is $G = \omega G$, is equivalent to

$$(p_\alpha, G) = (p_\alpha, \omega G) = (\omega p_\alpha, G) = (-1)^{|\alpha| - \ell(\alpha)}(p_\alpha, G)$$

(since $\omega p_k = (-1)^{k-1} p_k$), from which the assertion follows.

Now, the staircase S-functions belong to the subalgebra generated by the odd power sums. This follows for example from the Murnaghan–Nakayama rule, since the corresponding diagram has only odd hook lengths. In our formalism this amounts to considering the partial derivatives

$$\frac{\partial}{\partial p_k} s_{(n,n-1,\ldots,1)} = \frac{1}{k}\left[s_{(n-k,n-1,\ldots,1)} + s_{(n,n-1-k,\ldots,1)} + \cdots\right]$$

and applying the modification rules.

This subalgebra is of considerable interest as it possesses as linear basis the Schur Q-functions (which correspond to the spin characters of the symmetric group). But the only Schur functions s_λ contained in this algebra are the staircase functions, the staircase partitions $(n, n-1, \ldots, 1)$ being the only partitions equal to their 2-cores (cf James and Kerber 1981).

The 'if' part of the theorem follows from the next lemma:

Lemma 6.2. Let F be a symmetric function contained in the subalgebra of Q-functions and let G be self-associated. Then H given by $\langle F\rangle * \langle G\rangle = \langle H\rangle$ is self-associated.

To prove this, consider Murnaghan's formula for the inner tensor product (11). As $D_{\sigma_1} p_\alpha = \prod_j (p_{\alpha_j} + 1)$ and

$$D_{p_\alpha} = c\partial_{p_{\alpha_1}} \partial_{p_{\alpha_2}} \cdots$$

for some scalar c, the above considerations show that only those α with odd parts will contribute. But then $D_{p_\alpha} F$ still belongs to the algebra of Q-Functions and $D_{p_\alpha} G$ remains self-associated. As the product of Q-functions with a self-associated S-function remains self-associated, the lemma follows.

It remains to prove the 'only if' direction of the theorem.

So let λ and μ be partitions of m and n, respectively. If one of the partitions is not self-conjugate, say μ, then $s_\lambda s_\mu$, the summand of degree $m+n$ in Murnaghan's formula, contains a term $p_1^m p_\alpha$ with $|\alpha| - \ell(\alpha)$ odd. Hence the result cannot be self-associated. If both partitions are self-associated, but none of them is a staircase, then they differ from their 2-cores. Hence s_λ contains a term $p_1^a p_2^b$, $b > 0$ maximal, and s_μ a term $p_1^u p_2^v$, $v > 0$ maximal. Because of self-associativity, b and v are even and $p_1^{a+2} p_2^{b-1}$, $p_1^{u+2} p_2^{v-1}$ are not contained in the respective S-functions.

Consider again Murnaghan's formula (11). We show that the sum on the right-hand side contains a term $p_1^{a+u} p_2^{b+v-1}$. Because of degree considerations this can only happen for $\alpha = (2), (1,1), (1)$. But $\alpha = (1,1), (1)$ cannot contribute, for this would contradict the self-associativity. So the case $\alpha = (2)$ remains and the contribution is

$$(D_{p_2} p_1^a p_2^b D_{p_2} p_1^u p_2^v) p_2/2 = bv\ p_1^{a+u} p_2^{b+v-1}$$

which is non-zero because of the choice of a, b, u, v. So in this case again, H cannot be self associated.

7. Reduced inner plethysms

The concept of inner plethysms of S-functions was developed by Littlewood (1958a) who sketched some methods for their evaluation. A systematic procedure was given by Butler (1970) with further developments by King (1974). (For a modern account of inner plethysm in a Hopf-algebra formalism see Scharf and Thibon (1993).) A short tabulation has been given by Vanagas (1971). Most of the preceding results stem from Littlewood's observation that

$$\{m-1, 1\} \odot \{1^k\} = \{m-k, 1^k\} \tag{20}$$

which in reduced notation becomes

$$\langle 1\rangle \otimes \{1^k\} = \langle 1^k\rangle. \tag{21}$$

Littlewood's basic result can be extended to other simple cases such as

$$\langle 1\rangle \otimes \{21^k\} = \langle 21^k\rangle + \langle 21^{k-1}\rangle + \langle 1^{k+1}\rangle + \langle 1^k\rangle$$

and

$$\langle 1\rangle \otimes \{31^k\} = \langle 2\rangle\langle 1^{k+1}\rangle - \langle 21^{k+1}\rangle + \langle 21^{k-1}\rangle + \langle 1^{k+1}\rangle + \langle 1^k\rangle.$$

However, generalizations to other single hook reduced partitions do not seem to yield such simple expressions. An arbitrary S-function $\{\nu\}$ may always be expanded as a sum of elementary symmetric functions e_ρ and hence as sums of products of S-functions of the type $\{1^k\}$. Thus any plethysm of the type $\langle 1\rangle \otimes \{\mu\}$ can always be reduced to sums of products of reduced Kronecker products of the type $\langle 1^k\rangle * \langle 1^\ell\rangle \ldots$ (cf Salam and Wybourne 1989). The highest weight partition contained in $\langle 1\rangle \otimes \{\lambda\}$ is necessarily (λ) and thus

$$\langle 1\rangle \otimes \{\lambda\} \supset \langle\lambda\rangle + \cdots \tag{22}$$

and hence any reduced irreducible representation $\langle\lambda\rangle$ may be uniquely expanded as

$$\langle\lambda\rangle \equiv \langle 1\rangle \otimes \sum_\nu c^\nu_\lambda\{\nu\}. \tag{23}$$

For example, one can readily establish that

$$\begin{aligned}
\langle 2\rangle &= \langle 1\rangle \otimes (\{2\} - \{1\} - \{0\})\\
\langle 3\rangle &= \langle 1\rangle \otimes (\{3\} - \{2\} - \{1^2\} - \{1\})\\
\langle 21\rangle &= \langle 1\rangle \otimes (\{21\} - \{2\} - \{1^2\} + \{0\})\\
\langle 4\rangle &= \langle 1\rangle \otimes (\{4\} - \{3\} - \{21\} - \{2\} + \{1^2\} + \{1\})\\
\langle 31\rangle &= \langle 1\rangle \otimes (\{31\} - \{3\} - 2\{21\} + \{2\} - \{1^3\} + \{1\})\\
\langle 2^2\rangle &= \langle 1\rangle \otimes (-\{3\} + \{2^2\} - \{21\} + 2\{1^2\} + 2\{1\})\\
\langle 21^2\rangle &= \langle 1\rangle \otimes (\{21^2\} - \{21\} + \{2\} - \{1^3\} - \{0\}).
\end{aligned} \tag{24}$$

Remarkably, the expansion $\langle n\rangle = \langle 1\rangle \otimes \mathcal{L}(n)$ involves a list $\mathcal{L}(n)$ of S-functions that is multiplicity free and involves only single hook partitions. Indeed, we shall shortly show that

$$\mathcal{L}(n) = \sum_{a=1}^{n} (-1)^{[(n+a+\delta)/2]}(\{a1^{[(n-a)/2]}\} + \{a1^{[(n-a)/2]-1}\}). \tag{25}$$

It follows that any reduced inner plethysm $\langle\lambda\rangle \otimes \{\mu\}$ may be reduced to the form

$$\langle\lambda\rangle \otimes \{\mu\} = (\langle 1\rangle \otimes \sum_\nu c^\nu_\lambda\{\nu\}) \otimes \{\mu\} = \langle 1\rangle \otimes \left(\left(\sum_\nu c^\nu_\lambda\{\nu\}\right) \otimes \{\mu\}\right) \tag{26}$$

where $\delta = 2$ if *odd n*, otherwise zero; which gives another method of evaluating reduced inner plethysms. It follows from equations (25) and (26) that

$$\langle 2\rangle \otimes \{\lambda\} = \langle 1\rangle \otimes \sum_{\nu}(-1)^{|\mu|}(\{2\} \otimes \{\lambda/\mu\}) \cdot \{\tilde{\mu}/M\} \tag{27a}$$

$$\langle 21\rangle \otimes \{\lambda\} = \langle 1\rangle \otimes \sum_{\mu,\rho}(-1)^{|\mu|}\{21\} \otimes \{\lambda/\mu M\} \cdot \{\tilde{\mu}\} * \{\rho\} \cdot \{\rho\} \tag{27b}$$

where

$$M = \sum_{m=0}^{\infty}\{m\} = \sigma_1.$$

The above two results contain a phase factor showing that the method involves considerable overcounting. Specific calculation gives

$$\begin{aligned}\langle 2\rangle \otimes \{21\} = {} & \langle 51\rangle + \langle 5\rangle + \langle 42\rangle + 3\langle 41\rangle + 3\langle 4\rangle + \langle 321\rangle + 3\langle 32\rangle + 3\langle 31^2\rangle + 7\langle 31\rangle + 4\langle 3\rangle \\ & + 2\langle 2^2 1\rangle + 5\langle 2^2\rangle + \langle 21^3\rangle + 5\langle 21^2\rangle + 9\langle 21\rangle + 5\langle 2\rangle + \langle 1^4\rangle + 3\langle 1^3\rangle + 4\langle 1^2\rangle + 2\langle 1\rangle\end{aligned} \tag{28}$$

$$\begin{aligned}\langle 21\rangle \otimes \{21\} = {} & \langle 71\rangle + 2\langle 7\rangle + \langle 621\rangle + 5\langle 62\rangle + 5\langle 61^2\rangle + 17\langle 61\rangle + 14\langle 6\rangle + \langle 54\rangle + 2\langle 531\rangle + 9\langle 53\rangle \\ & + \langle 52^2\rangle + 2\langle 521^2\rangle + 20\langle 521\rangle + 45\langle 52\rangle + \langle 51^4\rangle + 10\langle 51^3\rangle + 47\langle 51^2\rangle + 81\langle 51\rangle \\ & + 45\langle 5\rangle + \langle 4^2 1\rangle + 5\langle 4^2\rangle + 3\langle 432\rangle + 3\langle 431^2\rangle + 25\langle 431\rangle + 47\langle 43\rangle + 3\langle 42^2 1\rangle \\ & + 20\langle 42^2\rangle + 2\langle 421^3\rangle + 30\langle 421^2\rangle + 118\langle 421\rangle + 149\langle 42\rangle + 10\langle 41^4\rangle + 64\langle 41^3\rangle \\ & + 163\langle 41^2\rangle + 185\langle 41\rangle + 78\langle 4\rangle + 3\langle 3^2 21\rangle + 16\langle 3^2 2\rangle + \langle 3^2 1^3\rangle + 20\langle 3^2 1^2\rangle \\ & + 73\langle 3^2 1\rangle + 82\langle 3^2\rangle + \langle 32^3\rangle + 2\langle 32^2 1^2\rangle + 25\langle 32^2 1\rangle + 73\langle 32^2\rangle + \langle 321^4\rangle + 20\langle 321^3\rangle \\ & + 118\langle 321^2\rangle + 270\langle 321\rangle + 235\langle 32\rangle + 5\langle 31^5\rangle + 47\langle 31^4\rangle + 163\langle 31^3\rangle + 280\langle 31^2\rangle \\ & + 240\langle 31\rangle + 83\langle 3\rangle + \langle 2^4 1\rangle + 5\langle 2^4\rangle + 9\langle 2^3 1^2\rangle + 47\langle 2^3 1\rangle + 82\langle 2^3\rangle + 5\langle 2^2 1^4\rangle \\ & + 45\langle 2^2 1^3\rangle + 149\langle 2^2 1^2\rangle + 235\langle 2^2 1\rangle + 162\langle 2^2\rangle + \langle 21^6\rangle + 17\langle 21^5\rangle + 81\langle 21^4\rangle \\ & + 185\langle 21^3\rangle + 240\langle 21^2\rangle + 173\langle 21\rangle + 55\langle 2\rangle + 2\langle 1^7\rangle + 14\langle 1^6\rangle + 45\langle 1^5\rangle + 78\langle 1^4\rangle \\ & + 83\langle 1^3\rangle + 55\langle 1^2\rangle + 19\langle 1\rangle + 2\langle 0\rangle.\end{aligned} \tag{29}$$

The above two results were extracted from a table of all reduced inner plethysms where the product of the weights of the two partitions are $\leqslant 10$. Copies of these tables are available as a TEXfile distributed via e-mail (bgw@risc.phys.torun.edu.pl). Two observations are immediately apparent. First, and not surprisingly, the multiplicities rapidly become very large, much more rapidly than for outer plethysms of the same weight. Second, the inner plethysm $\langle 21\rangle \otimes \{21\}$ is clearly self-associated, which at first sight is surprising. In that case both partitions defining the plethysm are staircase partitions. Indeed we shall shortly show that the necessary condition for a reduced inner plethysm $\langle\lambda\rangle \otimes \{\mu\}$ to be self-associated is that the partitions (λ) and (μ) are staircase partitions but before that we remark upon the application of equation (22) to non-simple phase groups.

8. The non-simple phase groups of $\mathcal{S}_n$ and $\mathcal{A}_n$

We may denote a $3j$ symbol for a group $\mathcal{G}$ as $(\lambda\mu\nu)^m_{ijk}$, where m is a multiplicity label and the ijk are indices. The symmetry properties of the $3j$ symbol is determined by its behaviour under simultaneous permutation of the representations $\lambda\mu\nu$ and the indices ijk with respect to the group $\mathcal{S}_3$. A problem arises when the three representations in the $3j$ symbol are identical, e.g. $(\lambda\lambda\lambda)^m_{ijk}$. In this case

$$(\lambda\lambda\lambda)^m_{\pi(ijk)} = \sum_s D^{\nu_m}_{ms}(\pi)(\lambda\lambda\lambda)^s_{ijk} \tag{30}$$

where π is an element of $\mathcal{S}_3$ that acts on the indices ijk and $D^{\nu_m}(\pi)$ is a matrix representing the permutation π (Derome 1966, van Zanten and de Vries 1973, Butler 1974, King 1974). A group $\mathcal{G}$ will be a *simple phase group* if for every representation λ of the group

$$(\lambda\lambda\lambda)^m_{\pi(ijk)} = \epsilon_m(\lambda\lambda\lambda)^m_{ijk} \tag{31}$$

where $|\epsilon_r|^2 = 1$.

A group will be said to be a *non-simple phase group* if there exists a λ such that the identity irreducible representation occurs in the mixed symmetry part of the Kronecker cube of λ. In that case the matrix representing $\{21\}$ is a 2×2 matrix and a simple ± 1 phase choice is not possible. In the case of the symmetric groups it suffices to examine the reduced inner plethysm $\langle 21\rangle \otimes \{21\}$ given in equation (29). For a given value of n each partition is made up to weight n and all resulting non-standard S-functions made standard. Thus for $n = 5$ the standardization rules give

$$\{-361^2\} = -\{5\} \qquad \{-261\} = +\{5\} \qquad \{-16\} = -\{5\} \qquad \{50\} = +\{5\}$$

leading to

$$5\{-361^2\} + 17\{-261\} + 14\{-16\} + 2\{50\} = (-5 + 17 - 14 + 2)\{5\} = 0.$$

Continuing we can readily establish that $\mathcal{S}_n$ is a simple phase group for $n \leqslant 5$.

For $n = 6$ we find $\{321\} \otimes \{21\} \supset \{6\}$ and hence $\mathcal{S}_6$ is a non-simple phase group as already noted (King 1974). For $n > 6$ we have from equation (29) the stable result that

$$\{n-3, 21\} \otimes \{21\} \supset 2\{n\} \tag{32}$$

establishing that indeed every symmetric group $\mathcal{S}_n$ with $n \geqslant 6$ is a non-simple phase group.

Now consider the alternating group $\mathcal{A}_n$ which is a subgroup of $\mathcal{S}_n$. Under the restriction $\mathcal{S}_n \leftarrow \mathcal{A}_n$ the irreducible representations $\{n\}$ and $\{1^n\}$ both yield the identity irreducible representation of $\mathcal{A}_n$. Furthermore, the irreducible representation $\{n-3, 21\}$ of $\mathcal{S}_n$ remains irreducible for all $n \geqslant 7$, allowing us to conclude immediately that all the alternating groups $\mathcal{A}_n$ with $n \geqslant 7$ are non-simple phase groups. The $\mathcal{A}_n$ with $n \leqslant 5$ are certainly simple phase groups, leaving only the case of $n = 6$ to be considered. The irreducible representation $\{321\}$ of $\mathcal{S}_6$ decomposes into a pair of real irreducible representations $\{321\}_+$ and $\{321\}_-$ of $\mathcal{A}_6$. It follows from equation (29), standardized for $n = 6$, that for $\mathcal{A}_6$

$$(\{321\}_+ + \{321\}_-) \otimes \{21\} \supset 2\{6\}. \tag{33}$$

Expanding the plethysm we obtain

$$(\{321\}_+ \otimes \{21\}) + (\{321\}_- \otimes \{21\}) + (\{321\}_+ + \{321\}_-)\{321\}_+\{321\}_- \supset 2\{6\}. \tag{34}$$

But $\{321\}_+\{321\}_- \supset \{321\}_+ + \{321\}_-$ and hence we must conclude that

$$\{321\}_\pm \otimes \{21\} \not\supset \{6\}. \tag{35}$$

The remaining irreducible representations of $\mathcal{A}_6$ may readily be seen to be simple phase, leading to the conclusion that

The symmetric groups S_n with $n \leqslant 5$ and the alternating groups $\mathcal{A}_n$ with $n \leqslant 6$ are simple phase groups and all other values of n correspond to non-simple phase groups.

9. The expansion $\mathcal{L}(n)$ and the Butler–Boorman theorem

We now give a formal proof of the S-function series expansion of $\mathcal{L}(n)$, as described by equation (25). This expansion is reminescent of the Butler–Boorman theorem (Butler 1970, Boorman 1975) and we offer a short, and novel, proof of that theorem. The technique used can be applied in a variety of situations, and we illustrate it on two further examples: the stable analysis of the representation of S_n in the space of Lie polynomials; and the computation of branching rules from a continuous group to the symmetric group.

These computations are more conveniently carried out in the notation of Macdonald (1979) and Scharf and Thibon (1992). The inner plethysm of a symmetric function G by a symmetric function F is denoted by $\hat{F}(G)$ (instead of $G \odot F$ in Littlewood's notation). Examples of the correspondence are $\{\lambda\} \odot \{\mu\} = \hat{s}_\mu(s_\lambda)$ and $\langle\lambda\rangle \otimes \{\mu\} = \hat{s}_\mu(\Gamma_1 s_\lambda) = \hat{s}_\mu(\langle s_\lambda\rangle) = \hat{s}_\mu(\langle\lambda\rangle)$.

Concerning $\mathcal{L}(n)$, it is easier to solve first the corresponding problem for stable permutation characters, that is, to find F_n such that

$$\langle\!\langle h_n\rangle\!\rangle = \sigma_1 h_n = \hat{F}_n(\sigma_1 h_1).$$

The result for $\langle n\rangle$ will follow, since

$$\langle n\rangle = \sigma_1(s_{(n)} - s_{(n-1)}) = \langle\!\langle h_n\rangle\!\rangle - \langle\!\langle h_{n-1}\rangle\!\rangle.$$

We have the generating series

$$\sum_{n\geqslant 0} q^n \sigma_1 h_n = \sigma_1\sigma_q = \sigma_1((1+q)X).$$

Next, we exploit the λ-ring formalism to rewrite it as

$$\sigma_1\left(\frac{1-q^2}{1-q}X\right).$$

The reason for this transformation is that we have, on the other hand,

$$\sigma_1\left(\frac{1-y}{1-x}X\right) = \hat{\sigma}_x(\sigma_1 h_1) * \hat{\lambda}_{-y}(\sigma_1 h_1) \tag{36}$$

(Kirillov and Pak 1990, see also Thibon 1992) so that

$$\sigma_1\left(\frac{1-q^2}{1-q}X\right) = \sum_{i,j\geqslant 0} q^i(-q^2)^j(\widehat{h_i e_j})(\sigma_1 h_1)$$

and

$$F_n = \sum_{i+2j=n} (-1)^j h_i e_j.$$

Now, $\langle n\rangle = \sigma_1 h_n - \sigma_1 h_{n-1} = (\widehat{F_n - F_{n-1}})(\sigma_1 h_1)$, and to obtain the result in reduced notation, one just has to observe that $\hat{G}(\sigma_1 h_1) = \widehat{D_{\sigma_1} G}(\langle 1\rangle)$. Thus,

$$\mathcal{L}(n) = D_{\sigma_1}(F_n - F_{n-1}) \tag{37}$$

and to conclude the proof of formula (25), it only remains to notice that $D_{\sigma_1} e_j = e_j + e_{j-1}$, $D_{\sigma_1} h_i = h_i + h_{i-1} + \cdots + h_1 + 1$ and $h_i e_j = s_{(i,1^j)} + s_{(i+1,1^{j-1})}$. This is better illustrated on an example, so let us take $n = 10$. We have

$$F_{10} = h_{10} - h_8 e_1 + h_6 e_2 - h_4 e_3 + h_2 e_4 - e_5$$
$$F_9 = h_9 - h_7 e_1 + h_5 e_2 - h_3 e_3 + h_1 e_4$$

$$\begin{aligned}
\mathcal{L}(10) &= D_{\sigma_1}(F_{10} - F_9) \\
&= D_{\sigma_1}[(h_{10} - h_9) - (h_8 - h_7)e_1 + (h_6 - h_5)e_2 \\
&\quad - (h_4 - h_3)e_3 + (h_2 - h_1)e_4 - e_5] \\
&= h_{10} - h_8(e_1 + 1) + h_6(e_2 + e_1) - h_4(e_3 + e_2) \\
&\quad + h_2(e_4 + e_3) - (e_5 + e_4) \\
&= s_{10} - (s_9 + s_{81} + s_8) + (s_{71} + s_{611} + s_7 + s_{61}) \\
&\quad - (s_{511} + s_{4111} + s_{51} + s_{411}) + (s_{3111} + s_{21111} + s_{311} + s_{2111}) \\
&\quad - (s_{11111} + s_{1111})
\end{aligned}$$

which is indeed the result predicted by (25).

We now provide a simple proof of (23). As already observed, this has the consequence that any character of the symmetric group S_n can be expressed as an integral linear combination of irreducible characters indexed by hook partitions. This result is due to Butler (1970) and independently to Boorman (1975). In the language of λ-rings, it means that the representation ring $R(S_n)$ of the symmetric group is generated as a λ-ring by the single element $[n-1, 1]$ (or, which amounts to the same, by the class of the representation by permutation matrices $[n-1][1] = [n-1, 1] + [n]$). However, this ring is far from being freely generated by one of these elements: a given symmetric function F of weight n has many different expressions of the form $F = \check{G}(s_{(n-1,1)})$. This is due to the fact that $[n-1, 1]$ is $(n-1)$-dimensional, so that its kth exterior power vanishes for $k \geqslant n$. However, with stable characters or reduced notation, the expansion (23) becomes unique. The reason for this is given by (22), which is more easily established within the framework of stable permutation characters.

Lemma 9.1. Let μ be a partition of m. Then

$$\hat{h}_\mu(\sigma_1 h_1) = \sigma_1 \cdot [h_\mu + F_\mu]$$

where F_μ is a (non-homogeneous) symmetric function of weight strictly less than m.

Proof. By induction on the length $\ell(\mu)$. For $\ell(\mu) = 1$, one has the explicit formula (Littlewood 1958a, cf Thibon (1992) for this formulation)

$$\hat{h}_m(\sigma_1 h_1) = \sigma_1 \sum_{k_1+2k_2+\cdots+mk_m=m} h_{k_1} h_{k_2} \cdots h_{k_m} = \sigma_1[h_m + F_m]$$

where F_m is clearly of weight $< m$. Now, let $h_\nu = h_k h_\mu$. Then,

$$\hat{h}_\nu(\sigma_1 h_1) = [\sigma_1(h_\mu + F_\mu)] * [\sigma_1(h_k + F_k)]$$

and using the expansion (10)

$$(\sigma_1 F) * (\sigma_1 G) = \sigma_1 \sum_{\alpha,\beta} (D_{u_\alpha} F)(D_{u_\beta} G)(v_\alpha * v_\beta)$$

where (u_α) and (v_β) are adjoint bases of symmetric functions, one sees that

$$\begin{aligned}\hat{h}_\nu(\sigma_1 h_1) &= \sigma_1 \left[h_\mu h_k + \sum_{|\alpha|+|\beta|\geqslant 1} D_{s_\alpha}(h_\mu + F_\mu) D_{s_\beta}(h_k + F_k)(s_\alpha * s_\beta) \right] \\ &= \sigma_1 [h_\nu + F_\nu]\end{aligned}$$

with F_ν of weight $\leqslant |\mu| + k - 1$, as required.

Now, by induction on $|\mu|$, we see that any series $\sigma_1 h_\mu$ can be (uniquely) written as linear combination with integer coefficients of series of the form $\hat{h}_\lambda(\sigma_1 h_1)$. As already pointed out, $\hat{G}(\sigma_1 h_1) = \widehat{D_{\sigma_1} G}(\langle 1 \rangle)$, so that this proves as well that any stable character $\langle F \rangle$ has a unique expression of the form $\langle F \rangle = \hat{G}(\langle 1 \rangle)$.

The above proof provides a simple algorithm for expressing the stable permutation characters in the form $\sigma_1 h_\lambda = \hat{G}_\lambda(\sigma_1 h_1)$. The first examples are, denoting for short $\hat{f}(\sigma_1 h_1)$ by $[f]$

$$\begin{aligned}
\langle\langle h_2 \rangle\rangle &= [h_2 - h_1] \\
\langle\langle h_{11} \rangle\rangle &= [h_{11} - h_1] \\
\langle\langle h_3 \rangle\rangle &= [h_3 - h_{11}] \\
\langle\langle h_{21} \rangle\rangle &= [h_{21} - 2h_{11} + h_1] \\
\langle\langle h_{111} \rangle\rangle &= [h_{111} + 3h_{11} + h_1] \\
\langle\langle h_4 \rangle\rangle &= [h_4 - h_{21}] \\
\langle\langle h_{31} \rangle\rangle &= [h_{31} - h_{111} + 4h_1] \\
\langle\langle h_{22} \rangle\rangle &= [h_{22} - 2h_{21} - h_{111} + 4h_{11} - h_2 + 3h_1] \\
\langle\langle h_{211} \rangle\rangle &= [h_{211} - h_{21} - 3h_{111} + 9h_{11} + 8h_1] \\
\langle\langle h_{1111} \rangle\rangle &= [h_{1111} - 6h_{111} + 11h_{11} + 18h_1].
\end{aligned}$$

There is an interesting formula for $[h_{(1^m)}] = (\sigma_1 h_1)^{*m}$, that is

$$[h_{(1^m)}] = \sigma_1 \sum_{k=1}^{m} S(m,k) h_{(1^k)}$$

where the $S(m,k)$ are the Stirling numbers of the second kind. This formula (which is equivalent to (5.5.19) in James and Kerber (1981)) is easily established by induction using formula (10) with $u_\lambda = p_\lambda$. Another easy but useful formula for computing with inner plethysms is

$$\hat{p}_k(\sigma_1 h_1) = \sigma_1 \sum_{d|k} p_d. \tag{38}$$

These identities can, for example, be applied to the computation of the character of the representation of the symmetric group in a free Lie algebra. That is, let $A = \{a_1, \ldots, a_n\}$ be a finite set of non-commuting indeterminates, let $L(A)$ be the free Lie algebra generated by A in the algebra of non-commutative polynomials $\mathbb{C}\langle A\rangle$, and let the action of S_n be defined by $\pi(a_i) = a_{\pi(i)}$ extended to an automorphism of $\mathbb{C}\langle A\rangle$. The character of the representation of $GL(n)$ in the homogeneous component $L_m(A)$ of degree m is well known (cf Reutenauer (1993)): it is the symmetric function

$$\ell_m = \frac{1}{m}\sum_{d|m}\mu(d)p_d^{m/d} \tag{39}$$

where μ is the Moebius function. The characteristic of the action of S_n on $L_m(A)$ is thus given by the inner plethysm $\hat{\ell}_m(h_1h_{n-1})$, which is the term of weight n in the series $\hat{\ell}_m(\sigma_1 h_1)$. These plethysms are easily computed by hand up to weight 7 or 8 (in the basis of stable permutation characters) using the above formulae. The first ones are

$$[\ell_2] = [e_2] = \langle\langle h_{11} - h_1\rangle\rangle$$

$$[\ell_3] = \langle\langle h_{21} - h_3 + h_{11}\rangle\rangle$$

$$[\ell_4] = \langle\langle h_{211} - h_{22} + 2h_{111} - h_{21} + 2h_{11} - h_2\rangle\rangle$$

$$[\ell_5] = \langle\langle h_{2111} + h_{221} - h_{311} + h_{32} + h_{41} - h_5 + 2h_{1111} + 5h_{111} + 3h_{11}\rangle\rangle.$$

These expressions are then readily converted into stable irreducible characters by applying the operator D_{σ_1} and expressing the result in the basis of Schur functions. For example, with $m = 4$,

$$D_{\sigma_1}[h_{211} - h_{22} + 2h_{111} - h_{21} + 2h_{11} - h_2] = 2 + 8s_1 + 6s_2 + 9s_{11} + 2s_3 + 5s_{21} + 3s_{111} + s_{31} + s_{211}$$

which amounts to saying that the character of the symmetric group in the space of Lie polynomials of degree 4 is, in reduced notation,

$$2\langle 0\rangle + 8\langle 1\rangle + 6\langle 2\rangle + 9\langle 11\rangle + 2\langle 3\rangle + 5\langle 21\rangle + 3\langle 111\rangle + \langle 31\rangle + \langle 211\rangle.$$

As a last illustration, we will show how to derive the branching rule for $O(n) \supset S_n$ (Salam and Wybourne 1989) by means of the stable character formalism. Using the orthogonal Schur functions (Littlewood 1950)

$$o_\lambda = D_{\lambda_{-1}(h_2)}s_\lambda \tag{40}$$

where $\lambda_{-1}(h_2) = \sum_r(-1)^r e_r \circ h_2 = \sum_r(-1)^r\{2\} \otimes \{1^r\}$, one has to compute the inner plethysm $\hat{o}_\lambda(h_1h_{n-1})$. Working with the generating series $\sigma_1 h_1$ and the reciprocity formula (cf Scharf and Thibon 1993)

$$(\hat{F}(\sigma_1 h_1), G) = (F, G(\sigma_1)) \tag{41}$$

we have, F being any symmetric function,

$$(\hat{o}_\lambda(\sigma_1 h_1), F) = (o_\lambda, F(\sigma_1)) = (D_{\sigma_1}D_{\lambda_{-1}}D_{\lambda_{-1}(h_2)}s_\lambda, F(\sigma_1))$$

$$= ((D_{\sigma_1}\widehat{D_{\lambda_{-1}\cdot\lambda_{-1}(h_2)}s_\lambda})(\sigma_1 h_1), F) = (\widehat{D_{\lambda_{-1}\cdot\lambda_{-1}(h_2)}}s_\lambda(\langle 1\rangle), F)$$

so that

$$\hat{o}_\lambda(\sigma_1 h_1) = (\widehat{D_G s_\lambda})(\langle 1\rangle) \tag{42}$$

where (cf Littlewood 1950, Macdonald 1979)

$$G = \lambda_{-1}\cdot\lambda_{-1}(h_2) = \prod_i(1 - x_i)\prod_{i\leqslant j}(1 - x_ix_j) = \sum_\epsilon(-1)^{(|\epsilon| - \ell(\epsilon))/2}s_\epsilon \tag{43}$$

where the sum is over all self-conjugate partitions.

10. Staircase S-functions

The S-functions indexed by staircase partitions $\tau(n) = (n, n-1, \ldots)$ possess a number of interesting properties (cf King *et al* 1981, King and Wybourne 1982) that have made themselves evident in the present work. In particular we now prove the self-associativity of inner plethysms involving staircase partitions.

As already remarked in section 6 staircase functions are the only Schur functions contained in the algebra generated by the odd power sums. If we denote this subalgebra by J and the ideal generated by the even power sums $p_2, p_4, \ldots$ by I, then we have the direct sum decomposition $I \oplus J$ of symmetric functions. This section is devoted to the proof of the following lemma.

Lemma 10.1. Let F, G be two symmetric functions contained in the algebra generated by the odd powers sums. Then H given by $\langle F\rangle \otimes G = \langle H\rangle$ is also in this algebra.

Proof. Because of an argument similar to that of lemma 6.2, it suffices to consider the case $G := p_m$, m odd. That means that we will have to consider the action of the so-called Adams operator $\hat{p}_m$ of inner plethysm. Its adjoint $\hat{\phi}_m$ with respect to the canonical scalar product is multiplicative with respect to the Littlewood–Richardson multiplication and obeys

$$\hat{\phi}_m p_\alpha = \prod_i p_{\gcd(\alpha_i,m)}^{\alpha_i/\gcd(\alpha_i,m)}$$

where 'gcd' means greatest common divisor. Now we have

$$(\hat{p}_m(\langle F\rangle), U) = (D_{\lambda_{-1}}F, D_{\sigma_1}\hat{\phi}_m D_{\lambda_{-1}}(D_{\sigma_1}U)). \tag{44}$$

The operator $\Phi := D_{\sigma_1}\hat{\phi}_m D_{\lambda_{-1}}$ is clearly a homomorphism and it follows by a simple computation that $\Phi(I) \subseteq I$, $\Phi(J) \subseteq J$. As a consequence of the orthogonal decomposition, the adjoint Ψ of Φ also has this property and we may rewrite equation (44) as

$$(\sigma_1 \Psi D_{\lambda_{-1}} F, U) = (\langle D_{\sigma_1}\Psi D_{\lambda_{-1}} F\rangle, U).$$

But F is in I and so the symmetric function in the brackets $\langle - \rangle$. This proves 10.1. $\square$

A direct consequence is the following corollary.

Corollary 10.2. If $\langle\lambda\rangle \otimes \{\mu\} = \langle H\rangle$, where λ and μ are staircase partitions, then H is self-associated.

11. Concluding remarks

The reduced notation outlined herein gives a powerful tool for investigating the stability of diverse properties of the symmetric group such as Kronecker products, irreducible characters, inner plethysms etc. A number of hitherto unnoticed features of these properties have been exposed and formal proofs developed. The computational aspects of this paper were made using the program SCHUR†, indeed it was this program that led to the initial conjectures that, in turn, led to the content of this paper.

† SCHUR is an interactive program for calculating the properties of Lie groups and symmetric functions by Brian G Wybourne, distributed by S Christensen, PO Box 16175, Chapel Hill, NC 27516 USA. Email: stevec@wri.com

References

Boorman E H 1975 *Trans. Am. Math. Soc.* **205** 127–49
Butler P H 1970 *J. Physique Coll.* **31** C4 47–50
—— 1974 *Phil. Trans. R. Soc.* A **277** 545–85
Butler P H and King R C 1973 *J. Math. Phys.* **14** 1176–83
Carré C and Thibon J-Y 1992 *Adv. Appl. Math.* **13** 390–403
Derome J R 1966 *J. Math. Phys.* **8** 714–16
Frobenius F G 1900 *Sitzungsber. Kgl. Preuss. Akad. Wiss.* 516–34
—— 1904 *Sitzungsber. Kgl. Preuss. Akad. Wiss.* 558–71
Gamba A 1952 *Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **12** 167–9
Hoffman P 1989 *Adv. Math.* **74** 135–43
James G and Kerber A 1981 *The Representation Theory of the Symmetric Group* (Reading, MA: Addison-Wesley)
Jarvis P D and Yung C M 1993 *J. Phys. A: Math. Gen.* **26** 1881–900
Jing N 1991a *J. Alg.* **138** 340–98
—— 1991b *Adv. Math.* **87** 226–49
Kac V G 1990 *Infinite Dimensional Lie Algebras* (Cambridge: Cambridge University Press)
Kerber A 1992 *Algebraic Combinatorics via Finite Group Actions* (Mannheim: BI-Wissenschaftsverlag)
King R C 1974 *J. Math. Phys.* **15** 258–67
King R C and Wybourne B G 1982 *J. Phys. A: Math. Gen.* **15** 1137–41
King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509–38
Kirillov A A and Pak I M 1990 *Funct. Anal. Appl.* **24** 172–6
Knutson D 1972 *Springer Lect. Notes in Math.* **308**
Kretzschmar M 1960a *Z. Phys.* **157** 433–56
—— 1960b *Z. Phys.* **158** 284–303
Lascoux A and Pragacz P 1988 *J. Phys. A: Math. Gen.* **21** 4105–14
Littlewood D E 1950 *The Theory of Group Characters* 2nd edn (Oxford: Clarendon)
—— 1958a *Can. J. Math.* **10** 1–16
—— 1958b *Can. J. Math.* **10** 17–32
Littlewood D E and Richardson A R 1934 *Phil. Trans. R. Soc.* A **233** 99–141
Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 327–48
Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Oxford University Press)
Morris A O 1963a *Quart. J. Math.* **14** 247–53
—— 1963b *Math. Z.* **81** 112–23
Murnaghan F D 1938 *Am. J. Math.* **60** 761–84
—— 1951 *Proc. N.A.S. (USA)* **37** 55–8
—— 1955 *Proc. N.A.S. (USA)* **41** 396–8
Remmel J B 1989 *J. Alg.* **120** 100–18
Remmel J B and Whitehead T 1993 On the Kronecker product of Schur functions of two row shapes *Preprint*
Reutenauer C 1993 *Free Lie Algebras* (Oxford: Oxford University Press)
Salam M A and Wybourne B G 1989 *J. Phys. A: Math. Gen.* **22** 3771–8
—— 1991 *J. Phys. A: Math. Gen.* **24** L317–21
—— 1992 *J. Phys. A: Math. Gen.* **25** 2297–310
Scharf T and Thibon J-Y 1992 A Hopf algebra approach to inner plethysm *LITP internal report* 92.72 (to appear in *Adv. Math.*)
Specht W 1960 *Math. Z.* **73** 312–29
Thibon J-Y 1991 *J. Alg. Comput.* **1** 207–21
—— 1992 *Bayreuther Math. Schr.* **40** 177–201
Vanagas V V 1971 *Algebraic Methods in Nuclear Theory* (Vilnius: Mintis) (in Russian)
Wagner B 1979 *Bayreuther Math. Schr.* **2**
van Zanten A J and deVries E 1973 *J. Alg.* **25** 475